Note that $\lambda^n(1,2) = (s_n, s_{n+1})$ and $\lambda^n(1,6) = (t_n, t_{n+1})$ for all $n \geq 0$.

Since $(1,2)$ and $(1,6) \in S$ we see that $(a,b) \in S$ for any $\{a,b\} = \{s_n, s_{n+1}\}$ or $\{a,b\} = \{t_n, t_{n+1}\}$ and $n \geq 0$ by (5).

Now suppose $(a,b) \in S$. Since $(b,a) \in S$ as well, we can suppose without loss of generality that $a < b$. By 5) and 7) there exists an integer $d \geq 0$ such that $\rho^d(a,b) = (a^*, b^*)$ with $a^* < b^* \leq 10$. By (6) we must have $\rho^d(a,b) = (1,2)$ or $\rho^d(a,b) = (1,6)$. Since $(a,b) = \lambda^d(\rho^d(a,b))$ we have $(a,b) = \lambda^d(1,2)$ or $(a,b) = \lambda^d(1,6)$.

Thus $ab-1$ divides $a^4 + 3a^2 + 1$ if and only if $a$ and $b$ are consecutive elements of either of the sequences $s_n$ or $t_n$ given above. Since the first few terms of $s_n$ are $1, 2, 15, 118, 929, 7314, 57583, \ldots$ and the first few terms of $t_n$ are $1, 6, 47, 370, 2913, 22934, 180559, \ldots$ the first few solutions to our problem (with $a \leq b$) are

$$(a,b) = (1,2), (2,15), (15,118), (118,929), (929,7314), (7314,57583), \ldots$$

and

$$(a,b) = (1,6), (6,47), (47,370), (370,2913), (2913,22934), (22934,180559), \ldots$$

**Also solved by Ed Gray, Highland Beach, FL; Kenneth Korbin, NewYork, NY; Toshihiro Shimizu, Kawasaki, Japan; Anna V. Tomova (three solutions), Varna, Bulgaria, and the proposer.**

- **5436:** *Proposed by Arkady Alt, San Jose, CA*

Find all values of the parameter $t$ for which the system of inequalities

$$\mathbf{A} = \begin{cases} \sqrt[4]{x+t} \geq 2y \\ \sqrt[4]{y+t} \geq 2z \\ \sqrt[4]{z+t} \geq 2x \end{cases}$$

**a)** has solutions;

**b)** has a unique solution.

**Solution by the Proposer**

**a)** Note that **(A)** $\Longleftrightarrow \begin{cases} t \geq 16y^4 - x \\ t \geq 16z^4 - y \\ t \geq 16x^4 - z \end{cases} \Longrightarrow 3t \geq 16y^4 - x + 16z^4 - y + 16x^4 - z =$

$\left(16x^4 - x\right) + \left(16y^4 - y\right) + \left(16z^4 - z\right) \geq 3 \min\limits_x \left(16x^4 - x\right) \Longrightarrow t \geq \min\limits_x \left(16x^4 - x\right).$

For $x \in \left(0, \dfrac{1}{16}\right)$, using the AM-GM Inequality, we obtain

$x - 16x^4 = x\left(1 - 16x^3\right) = \sqrt[3]{x^3\left(1-16x^3\right)^3} = \sqrt[3]{\dfrac{\left(48x^3\right)\left(1-16x^3\right)^3}{48}} \leq$

$\sqrt[3]{\dfrac{1}{48} \cdot \left(\dfrac{48x^3 + 3 - 3 \cdot 16x^3}{4}\right)^4} = \sqrt[3]{\dfrac{1}{48} \cdot \left(\dfrac{3}{4}\right)^4} = \dfrac{3}{16}$. And since $x - 16x^4 \leq 0$ for

$x \notin \left(0, \frac{1}{16}\right)$ , then for all $x$ the inequality $x - 16x^4 \le \frac{3}{16}$ holds. Since the upper bound is $\frac{3}{16}$ for values

$x - 16x^4$ is attainable when $x = \frac{1}{4}$ , then $\max\left(x - 16x^4\right) = \frac{3}{16} \iff$

$\min_x \left(16x^4 - x\right) = -\frac{3}{16}$.

Thus $t \ge -\frac{3}{16}$ is a necessary condition for the solvability of system **(A).**

Let's prove sufficiency.

Let $t \ge -\frac{3}{16}$. Since function $h(x)$ is continuous in $R$ and $\min_x \left(16x^4 - x\right) = -\frac{3}{16}$, then $\left[-\frac{3}{16}, \infty\right)$ is the range of $h(x)$ . This means that for any $t \ge -\frac{3}{16}$ the equation $16x^4 - x = t$

has solution in $R$ and since for any $u$ which is a solution of the equation $16x^4 - x = t$ the triple $(x, y, z) = (u, u, u, )$ is a solution of the system **(A)** then for such $t$ system **(A)** solvable as well.

**Remark.**

Actually the latest reasoning about the solvability of system **(A)** if $t \ge -\frac{3}{16}$ is redundant for **(a)** because suffices to note that for such $t$ the triple $(x, y, z) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ satisfies to **(A)**.

But for **(b)** criteria of solvability of equation $16x^4 - x = t$ in form of inequality $t \ge -\frac{3}{16}$ is important.

**b)** Note that system **(A)** always have more the one solution if $t > -\frac{3}{16}$.

Indeed, let for any $t_1, t_2 \in \left(-\frac{3}{16}, t\right)$ such that $t_1 \ne t_2$ equation $16u^4 - u = t_i$ has solution $u_i, i = 1, 2$.

Then $u_1 \ne u_2$ and two distinct triples $(u_1, u_1, u_1), (u_2, u_2, u_2)$ satisfy to the system **(A).**

Let $t = -\frac{3}{16}$.Then $-\frac{3}{16} \ge 16y^4 - x \implies -\frac{3}{16} + x - y \ge 16y^4 - y \ge -\frac{3}{16}$.

Hereof $x - y \ge 0 \iff x \ge y$. Similarly $-\frac{3}{16} \ge 16z^4 - y$ and $-\frac{3}{16} \ge 16x^4 - z$ implies $y \ge z$ and $z \ge x$, respectively. Thus in that case $x = y = z$ and all solutions of the system **(A)** are represented by solutions of one equation $16x^4 - x = -\frac{3}{16} \iff$

$16x^4 - x + \frac{3}{16} = 0 \iff 256x^4 - 16x + 3 = 0$ which has only root $\frac{1}{4}$ because

$256x^4 - 16x + 3 = (4x - 1)^2 \left(16x^2 + 8x + 3\right)$.

Thus, system **(A)** has unique solution iff $t = \frac{1}{4}$.

**Also solved by Ed Gray, Highland Beach,FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David Stone and John Hawkins, Georgia**